Indian Institute of Technology Goa

M. Tech. Thesis

Impossibility results for the sum of two Width-2 Invertible Arithmetic Branching Programs

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A thesis submitted in partial fulfilment of the requirements
for the degree of Master of Technology

in the

School of Mathematics and Computer Science



May 2022

Dedicated to my parents and my sister.

Declaration

I declare that this thesis was prepared by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Certificate of Approval

The examination committee has accepted this thesis entitled **Impossibility results for the sum of two Width-2 Invertible Arithmetic Branching Programs**, submitted by **Harsimran Singh** to the *Indian Institute of Technology Goa*, for the partial fulfillment of the requirements for the degree of *Master of Technology* in *Computer Science and Engineering*. The student has successfully defended the thesis held on May 30, 2022.

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Abstract

Impossibility results for the sum of two Width-2 Invertible Arithmetic Branching Programs

We consider the problem of proving the impossibility results for the sum of two width-2 invertible ABPs, i.e. the polynomials which are not computable as a sum of two width-2 invertible ABPs. In particular, we study the sum of two triangular ABPs, which are canonical forms of width-2 invertible ABPs. It has been conjectured that the polynomial $x_1x_2 + x_3x_4 + x_5x_6$ cannot be computed as the sum of two triangular ABPs. However, it has only been proved for the case where the linear forms occurring in the diagonal edge weights in the triangular ABPs are 3-wise independent. In this work, we look at the case where the linear forms are pairwise independent.

We argue that when the linear forms occurring in the diagonal edge weights are pairwise independent, then the multi-sets of diagonal edge weights in both the triangular ABPs are the same. Thereafter, we show that when the diagonal edge weights occur in the same sequence (or the reverse sequence) in both the triangular ABPs, then the sum of both the ABPs cannot compute $x_1x_2 + x_3x_4 + x_5x_6$.

Although we have not yet been successful in proving the conjecture for the case where the linear forms are pairwise independent, we hope that the ideas presented in this thesis and the results we got would be helpful in ultimately proving the conjecture.

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Acknowledgements

I am highly grateful to my supervisor, Dr. Arpita Korwar for giving me an opportunity to work with her. The meetings and discussions that I had with her helped me gain a lot of understanding and knowledge of various topics in Algebra and Computational Complexity. She provided me with a lot of insights into the thesis problem and was always available for clarifying my doubts. Her feedback and suggestions were very useful at every stage in the M.Tech project. This work would not have been possible without her ideas, patience and endless support.

I am thankful to the Computer Science and Engineering Department at IIT Goa for providing an intellectual environment. The faculty members possess a lot of wisdom and mastery over their respective domains. Their courses were up-to-date, well coordinated and carefully planned. I would also like to thank Aman, Rohit and Sakar for helping me to improve my programming and presentation skills.

I would like to express my gratitude to my parents and all the family members for their immense emotional support, affection and words of encouragement. I am indebted to them for their love and sacrifices.

I believe that soulful music provides peace and tranquility to the mind which helps to freshen it up for intellectual activities. For this reason, I thank Nusrat Fateh Ali Khan for his extraordinary qawallis which take me into great depths of emotions and new worlds full of ecstasy. I would also like to appreciate Trio Mandili, whose music videos were very helpful in going through challenging times. The beauty, talent and smiles of Tako, Tatuli and Mari leave a deep impression on the heart.

Finally, I thank the Akal Purakh Waheguru, who is the source of all strength. Without his will nothing comes to pass. He continues to bless and nourish the ungrateful and sinful people like me. I owe a lot to the Gurus, from Guru Nanak Dev Ji to Guru Granth Sahib Ji, whose Bani provides a connection with the Almighty Lord and gives direction, meaning and purpose to life.

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— 1 —

Introduction

For computing polynomials, one of the simplest and most well-known models are Arithmetic Circuits. At a high level, they are directed acyclic graphs having one or more input nodes and only one output node, which has no outgoing edges ([Val79]). The remaining nodes are either "$\times$" or "$+$" gates which output the product and sum of the inputs respectively. Input nodes are some constants or a polynomial over a field $\mathbb{F}$. The polynomial computed by an arithmetic circuit is computed at its output node. The circuit's depth is given by the length of a longest path from an input node to the output node. The number of edges in the circuit is said to be the size of the circuit. Valiant ([Val79]) defined the complexity class VP and its non-deterministic equivalent VNP for studying complexity of arithmetic circuits. VP and VNP are similar in nature to the complexity classes P and #P respectively, i.e. the role played by VP in Arithmetic Circuit Complexity is analogous to the role played by P in Computational Complexity. The class VP consists of polynomial families with polynomially bounded degree which can be computed by polynomial size arithmetic circuits ([San09]). VP $\neq$ VNP is an algebraic counterpart of P $\neq$ NP conjecture. Under generalized Riemann hypothesis ([Bür00]), P $\neq$ NP implies VP $\neq$ VNP. So, the most important question in Computational Complexity, P vs NP, boils down to finding lower bounds for polynomials in VNP.

Another useful model to represent polynomials are Arithmetic Branching Programs (ABPs). Informally, they are layered directed acyclic graphs with edge weights as linear polynomials over a field $\mathbb{F}$ (a formal definition is provided in section 2.1). The expressive power of ABPs is equal to the expressive power of the determinant of matrices with entries as linear polynomials ([MV97]). All the polynomials computable by arithmetic formulas can be computed by ABPs. Furthermore, every polynomial computable by an ABP can be computed by a polynomial sized arithmetic

circuit (For example, see [Raz05]). Hence, the computing power of an ABP is between the computing powers of arithmetic formulas and arithmetic circuits.

The number of vertices contained in the layer with the highest cardinality in the ABP is known as its width. This thesis mostly deals with width-2 ABPs. Saxena, Saha and Saptharishi ([SSS09]) proved that the Polynomial Identity Testing (PIT) of depth-3 arithmetic circuits can be reduced to Polynomial Identity Testing of width-2 commutative ABPs. The expressive power of an ABP heavily depends upon its width. Ben-Or and Cleve ([BC92]) showed that the family of width-3 ABPs can represent any formula, i.e. width-3 ABPs is a complete model. Allender and Wang ([AW16]) proved the existence of polynomials which are not computable by *any* width-2 ABP, regardless of its size. Hence, width-2 ABPs is an *incomplete* model. However, Bringmann et al. ([BIZ18]) showed that the closure of width-2 ABPs is complete.

ABPs can also be thought of as iterated matrix products with the entries of the matrices as linear polynomials (polynomials of degree ≤ 1) over the field $\mathbb{F}$. Invertible ABPs are special ABPs in which the determinant of every matrix in the iterated product is a non-zero field constant. Triangular ABPs (formally defined in section 2.2) are canonical forms of width-2 invertible ABPs. It has been proved that any polynomial computable by a width-2 invertible ABP is also computable by a triangular ABP ([Kor16]), which is comparatively easier to analyze. A polynomial whose highest degree homogeneous term doesn't factorize into linear forms (homogeneous degree one polynomials, or linear polynomials without a constant term) cannot be computed by a triangular ABP. Hence, it is impossible for a triangular ABP to compute the polynomial $x_1x_2 + x_3x_4$.

[Kor16] addressed the question of the impossibility results for the *sum* of two width-2 invertible ABPs. Impossibility results are the polynomials which cannot be computed by a model, regardless of its size. In [Kor16], partial progress was made towards proving that it is impossible to compute the polynomial $f = x_1x_2+x_3x_4+x_5x_6$ as a sum of two triangular ABPs. It was proved that any two triangular ABPs require a minimum of six layers for their sum to compute f. Furthermore, it was shown that if f can be computed as the sum of two triangular ABPs A and B, then the multi-sets of linear forms occurring in the weights of diagonal edges in both the triangular ABPs have to be the same. It was also proved that when these linear forms are 3-wise independent, then $A + B$ cannot compute the polynomial f.

Contribution of this thesis

In this work, we look at the case of pairwise independent linear forms, i.e. when any two linear forms chosen from the diagonal edge weights of the two triangular ABPs are linearly independent. It implies that the linear forms cannot be repeated and no linear form can be obtained by multiplying any other linear form with a field constant. We show that when $A + B = x_1x_2 + x_3x_4 + x_5x_6$ and the linear forms occurring in the weights of diagonal edges in both the triangular ABPs A and B are pairwise independent, then not only the multi-sets of linear forms, but the multi-sets of linear polynomials occurring as diagonal edge weights in both the ABPs are also the same. We proceed by comparing the degree$(n-1)$ homogeneous parts of both the ABPs, where n is the cardinality of the multi-set of linear forms. Since the sum of both the ABPs is of degree 2, the degree ≥ 3 parts of both the ABPs must cancel each other. Hence, the degree$(n-1)$ parts must be the same. Using this fact, we obtain that when S_n, the set consisting of n products of $n-1$ linear forms in the triangular ABPs, is linearly independent, then the multi-sets of linear polynomials have to be the same. Thereafter, we prove that S_n is linearly independent if and only if the linear forms are pairwise independent. Subsequently, we show that when the linear forms are pairwise independent and the linear polynomials occur in the same order (or the reverse order) in A and B, then A and B are essentially the same ABPs. Therefore, $A + B$ cannot compute $x_1x_2 + x_3x_4 + x_5x_6$.

Organization of this thesis

In chapter 2, we introduce the preliminary concepts. In particular, we define the notions of Invertible ABPs and Triangular ABPs. We also list some useful properties of Triangular ABPs. In chapter 3, we analyze the sum of two triangular ABPs. In this chapter, we derive the main results of our work. In chapter 4, we give concluding remarks along with some possible directions for future work.

— 2 —

Preliminaries

In this chapter, we form the necessary background for our work. We start by defining associates for linear polynomials. We shall use this concept later to prove the theorems in chapter 3.

Definition 2.1 (Associates)**.** Two linear polynomials L and L' over a field $\mathbb{F}$ are called associates if $L = cL'$ for some $c \in \mathbb{F} \setminus \{0\}$.

2.1 Arithmetic Branching Programs

Now, we take a look at Arithmetic Branching Programs, our main subject of study in this thesis.

Definition 2.2 (Arithmetic Branching Programs)**.** ([Nis91]) An Arithmetic Branching Program (ABP) is a layered directed acyclic graph with the first layer consisting of exactly one source vertex s and the last layer containing exactly one sink vertex t. Layers are numbered from 0 to n, where $n+1$ is the total number of layers. Edges only exist between successive layers i.e. if (x, y) is an edge from the vertex x to vertex y, then $x \in V_i$ and $y \in V_{i+1}$ for some $0 \leq i < n$, where V_k denotes the set of vertices in the layer number k. The weight of each edge is a linear polynomial over a field $\mathbb{F}$ in the set of variables $X = \{x_j \mid 1 \leq j \leq m\}$. The width of an ABP, w, is $\max_k\{|V_k|\}$ for $0 \leq k \leq n$.

In this thesis, we shall represent the weight of x by $\mathrm{W}(x)$, where x is a path or an edge in an ABP. We shall represent the weight of an edge connecting the two vertices v and v' in an ABP as $\mathrm{W}(v, v')$. If no edge exists between the two vertices v and v', then we take $\mathrm{W}(v, v') = 0$.

The weight of a path p is obtained by multiplying the weights of all the edges in the path p,

$$\mathrm{W}(p) = \prod_{e \in p} \mathrm{W}(e),$$

The polynomial computed at a vertex v, denoted by $\mathrm{f_v}(X)$, is given by the sum of the weights of all the paths from the source vertex s to the vertex v. Let P denotes the set of all the paths from s to the vertex v, then

$$\mathrm{f_v}(X) = \sum_{p \in P} \mathrm{W}(p),$$

The polynomial $\mathrm{f}(X)$ represented by an ABP is the polynomial computed at its sink vertex t.

Each layer of an ABP can be represented by a matrix. Hence, an ABP can also be viewed as iterated matrix multiplications $\prod_{i=1}^{n} M^{(i)}$, where $M^{(1)} \in \mathbb{F}[X]^{1 \times w}$, $M^{(i)} \in \mathbb{F}[X]^{w \times w}$ for $2 \leq i \leq n-1$ and $M^{(n)} \in \mathbb{F}[X]^{w \times 1}$. For this, we add dummy vertices in the intermediate layers having less than w vertices. These dummy vertices have no incoming or outgoing edges. Let the vertices in the layer number a, for $1 \leq a \leq n-1$, be denoted by $V_a = \{v_{a,b} \mid 1 \leq b \leq w\}$. The entries in the matrices are given by:

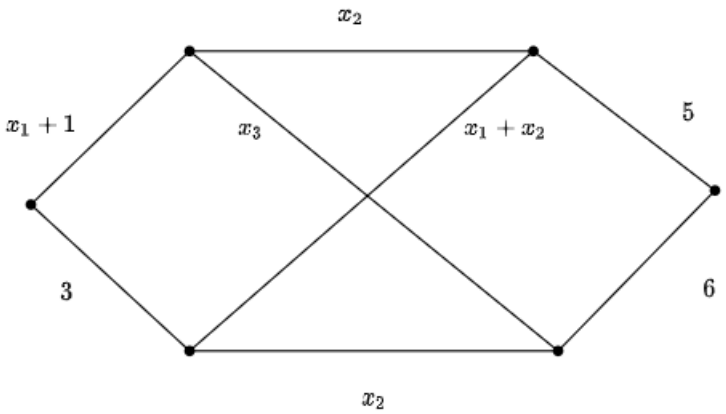
$$\begin{aligned}
M^{(1)}_{1,q} &= \mathrm{W}(s, v_{1,q}), \quad \text{for} \quad 1 \leq q \leq w, \\
M^{(i)}_{p,q} &= \mathrm{W}(v_{i-1,p}, v_{i,q}), \quad \text{for} \quad 1 \leq p, q \leq w \quad \text{and} \quad 2 \leq i \leq n-1, \\
M^{(n)}_{p,1} &= \mathrm{W}(v_{n-1,p}, t), \quad \text{for} \quad 1 \leq p \leq w.
\end{aligned}$$

The result of the matrix product $\prod_{i=1}^{n} M^{(i)}$ gives the polynomial computed by the ABP. The result of the product $\prod_{i=1}^{k} M^{(i)}$, for some $1 \leq k \leq n-1$, is a row matrix with w elements, whose entries represent the polynomials computed at the vertices in the layer number k. For instance, the polynomial computed at $v_{a,b}$ is the entry in the column number b in $\prod_{i=1}^{a} M^{(i)}$. Note that the polynomials computed at the dummy vertices are equal to zero.

Let $\mathrm{g_v}(X)$ be the polynomial obtained by summing up the weights of all the paths starting from the vertex v and ending at t, i.e. the polynomial computed by the sub-ABP (a subgraph of an ABP which is also an ABP) with v as the source vertex. Then, for any $1 \leq k \leq n-1$:

$$\mathrm{f}(X) = \sum_{v \in V_k} \mathrm{f_v}(X)\, \mathrm{g_v}(X).$$

Example 2.1. Consider the following ABP which computes $(x_1+1)(x_2)(5)+(x_1+1)(x_3)(6)+(3)(x_2)(6)+(3)(x_1+x_2)(5)$. It can be represented as the iterated matrix product $\begin{bmatrix} x_1+1 & 3 \end{bmatrix} \begin{bmatrix} x_2 & x_3 \\ x_1+x_2 & x_2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.



The left-most vertex is the source and the right-most vertex is the sink. Note that the edges in the ABP are directed and go from the vertices of a layer to the next layer. Hence, in this graph, the direction of all the edges is from left to right. Since the directions of the edges are implied from the definition of ABPs, we omit drawing them explicitly in the graphical representation for convenience.

2.2 Width-2 Arithmetic Branching Programs

This thesis mostly deals with width-2 ABPs, whose layers can be represented by 2×2 matrices.

2.2.1 Invertible Arithmetic Branching Programs

Allender and Wang ([AW16]) partitioned the set of all the matrices M with entries as linear polynomials into three types:

- **Inherently non-degenerate matrices:** The matrices whose determinant evaluates to a non-zero field constant, i.e. $|M| \in \mathbb{F} \setminus \{0\}$.
- **Inherently degenerate matrices:** The matrices with zero determinant, i.e. $|M| = 0$.

- **Potentially degenerate matrices:** The matrices whose determinant evaluates to a non-constant polynomial, i.e. $|M| \in \mathbb{F}[X] \setminus \mathbb{F}$.

Definition 2.3 (Invertible ABP)**.** If all the matrices representing the layers of an ABP are inherently non-degenerate, then the ABP is called as Invertible ABP.

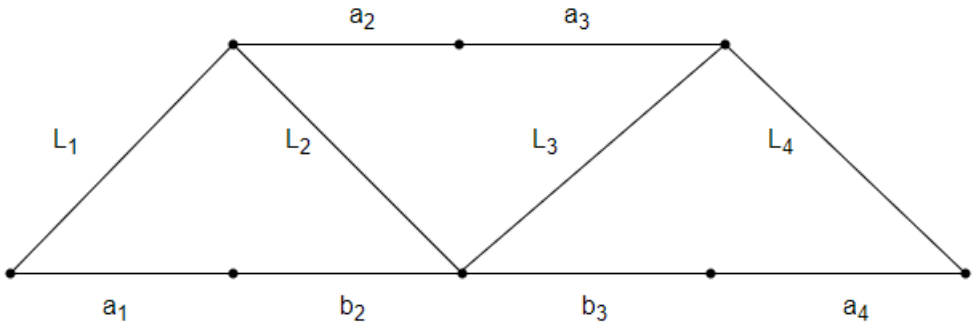
Computing power of width-2 invertible ABPs is equal to the computing power of triangular ABPs ([Kor16]). Hence, in this thesis, to study the sum of two width-2 invertible ABPs, we will study the sum of two triangular ABPs.

2.2.2 Triangular Arithmetic Branching Programs

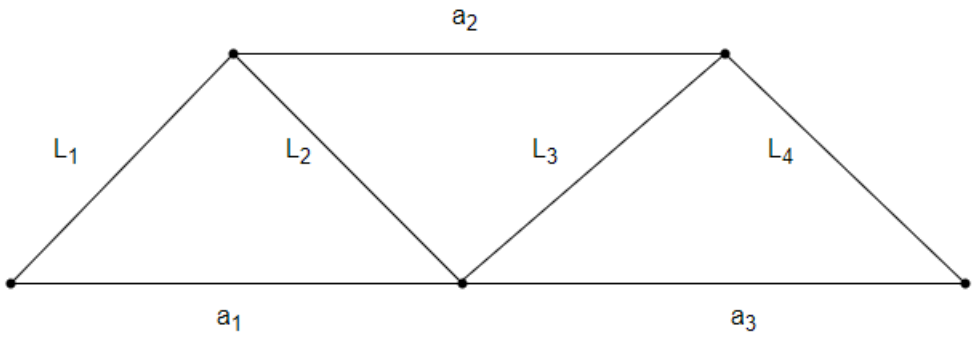
Now, we take a look at the canonical form of width-2 invertible ABPs, known as Triangular ABPs.

Definition 2.4 (Triangular ABP)**.** ([Kor16]) A triangular ABP is a width-2 ABP with intermediate layers of the form $\begin{pmatrix} a & L \\ 0 & b \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ L & b \end{pmatrix}$ alternately, where $a, b \in \mathbb{F} \setminus \{0\}$ and L is a polynomial of degree exactly one in $\mathbb{F}[X]$. The first layer is of the form $(L \quad a)$ and the last layer is of the form $\begin{pmatrix} L \\ a \end{pmatrix}$ or $\begin{pmatrix} a \\ L \end{pmatrix}$, where $a \in \mathbb{F} \setminus \{0\}$ and L is a polynomial of degree exactly one in $\mathbb{F}[X]$.

Example 2.2. Consider a triangular ABP representing the iterated matrix multiplication $\begin{bmatrix} L_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & L_2 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} a_3 & 0 \\ L_3 & b_3 \end{bmatrix} \begin{bmatrix} L_4 \\ a_4 \end{bmatrix}$:



In this ABP, we can merge the two edges joining the linear polynomials L_i and L_{i+1} for $1 \leq i \leq 3$ and by abuse of notation, rename the edge as a_i to get the following structure:



This simple structure allows us to understand the properties of triangular ABPs. Triangular ABPs derive their name from this graphical appearance. Hence, a triangular ABP can also be represented as an iterated matrix product with intermediate layers of the form $\left(\begin{smallmatrix} a & L \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ L & a \end{smallmatrix}\right)$ alternately.

2.2.2.1 Properties of Triangular ABPs

In a triangular ABP, since the weights of diagonal edges are polynomials of degree exactly one, we can write each diagonal edge weight L_i as $L_i = l_i + c_i$, where l_i is a linear form (homogeneous degree one polynomial) and $c_i \in \mathbb{F}$ is a field constant.
In this thesis, we use γ_A to represent the multiset of all the linear forms (which are denoted by l) occuring in the diagonal edge weights in the triangular ABP A and δ_A to represent the multiset of all the diagonal edge weights (which are linear polynomials denoted by L) in the triangular ABP A. For a general triangular ABP, they are denoted by γ and δ respectively.

Lemma 2.1. ([Kor16]) The highest degree homogeneous part of every polynomial computable by a triangular ABP factorizes into linear forms.

Proof. Let us consider a triangular ABP with n diagonal edges and the multi-set $\gamma = \{l_1, l_2, \ldots, l_n\}$.
Let P be the polynomial computed by the triangular ABP. It can be expressed in the following form:

$$P = \sum_{\gamma' \subseteq \gamma} k_{\gamma'} \prod_{l_i \in \gamma'} l_i, \quad \text{where} \quad k_{\gamma'} \in \mathbb{F},$$

The degree d homogeneous part of the polynomial P, denoted by $P_{[d]}$ is of the form:

$$P_{[d]} = \sum_{\gamma' \subseteq \gamma, |\gamma'| = d} k_{\gamma'} \prod_{l_i \in \gamma'} l_i, \quad \text{where} \quad k_{\gamma'} \in \mathbb{F},$$

Hence, the highest degree homogeneous part of P, which we denote by $\hat{P}$, is given by:

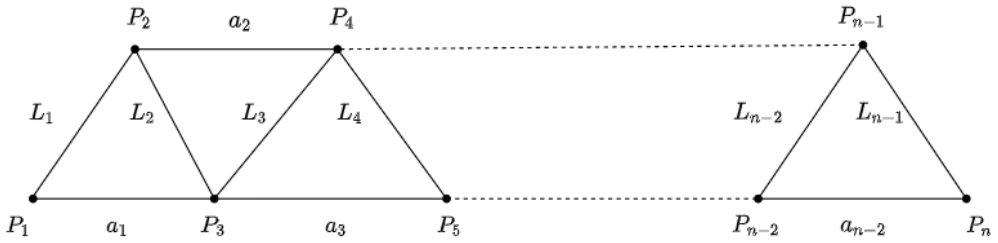
$$\hat{P} = \prod_{i \in [n]} l_i,$$

Therefore, we observe that $\hat{P}$ is a product of linear forms.

□

Since each diagonal edge weight in a triangular ABP is a non-constant linear polynomial, each diagonal edge weight contains a linear form. Hence, the size of the multi-sets γ and δ are equal. From the above proof, it is also clear that the degree of the polynomial P computed by the triangular ABP and the cardinality of γ are equal. Hence, $\deg(P) = |\delta| = |\gamma|$. Notice that in the proof of Lemma 2.1, $\hat{P} = P_{[n]}$.

Another interesting property of Triangular ABPs is that they can be used to compute the terms of the Fibonacci series. Let us assume that P_n is the polynomial computed at the n^{th} node from the left in a triangular ABP.



We can write P_n in terms of P_{n-1} and P_{n-2} as:

$$P_n = P_{n-1}L_{n-1} + P_{n-2}a_{n-2},$$

If we take P_1 to be 1 and all the edge weights, L_k $\forall\, 1 \leq k \leq n-1$ and a_k $\forall\, 1 \leq k \leq n-2$, as 1, then P_n will compute the n^{th} term of the Fibonacci series $1, 1, 2, 3, 5, 8 \ldots$.

— 3 —

Sum of two triangular ABPs

In this chapter, we look at the sum of two triangular ABPs in detail. Since the polynomial $x_1x_2 + c$, where c is a field constant, can be computed by a simple triangular ABP with just 2 layers, $x_1x_2 + x_3x_4$ can be easily expressed as a sum of two triangular ABPs. For example, the sum of the following two triangular ABPs computes $x_1x_2 + x_3x_4$.



In this thesis, we address the question whether the sum of two triangular ABPs can compute $x_1x_2 + x_3x_4 + x_5x_6$ or not. From Lemma 2.1, it is evident that $x_1x_2 + x_3x_4$ cannot be computed by a triangular ABP, since it doesn't factorize into linear forms. Therefore, if it is possible to compute $x_1x_2 + x_3x_4 + x_5x_6$ as the sum of two triangular ABPs A and B, then there must be degree ≥ 3 homogeneous parts in the polynomials computed by A and B. Moreover, these degree ≥ 3 homogeneous parts of both the polynomials must add up to zero.

The sum of two triangular ABPs can also be thought of as the difference because given any ABP B, it is easy to obtain an ABP B' which computes $-B$. Therefore, we wish to find out whether it is possible to give two triangular ABPs A and B in such a way that $A - B = x_1x_2 + x_3x_4 + x_5x_6$.

We begin by mentioning two important lemmas proved in [Kor16] which we will use later.

Lemma 3.1. ([Kor16]) *Let γ_A and γ_B be the multisets of linear forms occurring in the weights of diagonal edges in the triangular ABPs A and B respectively. If $A - B = x_1x_2 + x_3x_4 + x_5x_6$, then $\gamma_A = \gamma_B$.*

Lemma 3.2. ([Kor16]) *Let γ_A and γ_B be the multisets of linear forms occurring in the weights of diagonal edges in the two triangular ABPs A and B respectively. Let $\deg(A)$ and $\deg(B)$ be the degrees of the polynomials computed by A and B respectively. If $A - B = x_1x_2 + x_3x_4 + x_5x_6$ and $\deg(A) \geq 3$ or $\deg(B) \geq 3$, then $\deg(A) = \deg(B) = |\gamma_A| = |\gamma_B| \geq 6$.*

3.1 Comparison of linear polynomials in the two ABPs

In this section, we attempt to find out whether the multi-sets of linear *polynomials* occurring as diagonal edge weights in both the ABPs will be the same. If so, under what conditions?

Lemma 3.3. *Let A and B be two triangular ABPs in such a way that $A - B = x_1x_2 + x_3x_4 + x_5x_6$. Let the multi-sets of linear polynomials occurring as diagonal edge weights in A and B be $\delta_A = \{l_1 + c_1, l_2 + c_2, \ldots, l_n + c_n\}$ and $\delta_B = \{l_1 + d_1, l_2 + d_2, \ldots, l_n + d_n\}$ respectively. If the set of polynomials $S_n = \{\prod_{1 \leq i \neq j \leq n} l_i\}_{j=1}^{n}$ is linearly independent, then $c_i = d_i \ \forall \ 1 \leq i \leq n$, i.e. the multi-sets of linear polynomials occurring as diagonal edge weights in both the ABPs are the same.*

Proof. Let $A_{[d]}$ and $B_{[d]}$ denote the degree d homogeneous parts of the polynomials A and B respectively. Since $A - B = x_1x_2 + x_3x_4 + x_5x_6$, the degree ≥ 3 homogeneous parts of A and B must be equal to each other. Hence,

$$A_{[d]} = B_{[d]}, \quad \forall d \geq 3.$$

From Lemma 3.2, $n \geq 6$. Therefore,

$$\begin{aligned} A_{[n-1]} &= B_{[n-1]}, \\ \therefore \sum_{j=1}^{n} c_j \prod_{1 \leq i \neq j \leq n} l_i &= \sum_{j=1}^{n} d_j \prod_{1 \leq i \neq j \leq n} l_i, \\ \therefore \sum_{j=1}^{n} (c_j - d_j) \prod_{1 \leq i \neq j \leq n} l_i &= 0. \end{aligned}$$

Therefore, if $\{\prod_{1 \leq i \neq j \leq n} l_i\}_{j=1}^{n}$ is linearly independent, then $c_i = d_i \ \forall \ 1 \leq i \leq n$.

□

Now, an obvious question which arises is that when will the set $S_n = \{\prod_{1\leq i\neq j\leq n} l_i\}_{j=1}^{n}$ be linearly independent. It turns out to be linearly independent for the case where the linear forms in A and B are pairwise independent.

Lemma 3.4. *For a triangular ABP containing $n \geq 2$ diagonal edges, the set $S_n = \{\prod_{1\leq i\neq j\leq n} l_i\}_{j=1}^{n}$ is linearly independent if and only if $\gamma = \{l_i\}_{i=1}^{n}$, the set of linear forms occurring in the weights of diagonal edges in the ABP, is pairwise independent.*

Proof. Let's start by assuming that the set S_n is linearly independent. The set S_n consists of n polynomials of degree $n-1$ each. We can write the set as

$$S_n = \{t_1, t_2, \ldots, t_n\}, \quad \text{where} \quad t_j = \prod_{1\leq i\neq j\leq n} l_i = \frac{\prod_{i\in[n]} l_i}{l_j} \quad \text{for} \quad j = 1, 2, \ldots, n.$$

We want to prove that γ is pairwise independent. Let's assume that it is not the case. Therefore, we have two linear forms l_p and l_q in such a way that

$$l_p = cl_q, \quad c \in \mathbb{F} \setminus \{0\} \quad \text{and} \quad 1 \leq p \neq q \leq n.$$

Then, $t_q = ct_p$. But since S_n is linearly independent, this isn't possible. Therefore, we get a contradiction.

Conversely, let us assume that γ is pairwise independent. We want to show that S_n is linearly independent for any $n \geq 2$.
It can be shown by induction on n. When $n = 2$, the set $S_2 = \{l_1, l_2\}$ and $\gamma = \{l_1, l_2\}$. Hence, $S_2 = \gamma$ and therefore, S_2 is linearly independent.
Now, we assume the claim to be true for $n = k$, i.e., $\gamma = \{l_1, l_2, \ldots, l_k\}$ is pairwise independent and S_k is linearly independent. We want to prove it to be true for $n = k + 1$.
Let us choose a new linear form l_{k+1} such that our updated $\gamma = \{l_1, \ldots, l_k, l_{k+1}\}$ is pairwise independent. We want to prove that S_{k+1} is linearly independent.
The set S_k, containing k polynomials of degree $k-1$ each, can be written as,

$$S_k = \{t_1, t_2, \ldots, t_k\}, \quad \text{where} \quad t_j = \prod_{1\leq i\neq j\leq k} l_i, \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Similarly, S_{k+1} can be expressed as,

$$S_{k+1} = \{t'_1, t'_2, \ldots, t'_k, l_1 l_2 \ldots l_k\},$$
$$\text{where} \quad t'_j = \prod_{1\leq i\neq j\leq k+1} l_i = t_j.l_{k+1}, \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Let us assume that S_{k+1} is linearly dependent. Therefore, there exist $k+1$ field constants $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$, not all zero, such that

$$\alpha_1 t'_1 + \alpha_2 t'_2 + \cdots + \alpha_k t'_k + \alpha_{k+1} l_1 l_2 \ldots l_k = 0.$$

Depending upon the value of α_{k+1}, we have two possible cases.

Case 1 : $\alpha_{k+1} = 0$

When $\alpha_{k+1} = 0$,

$$\begin{aligned} &\alpha_1 t'_1 + \alpha_2 t'_2 + \cdots + \alpha_k t'_k = 0 \\ \implies &(\alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_k t_k).l_{k+1} = 0 \end{aligned}$$

By the induction hypothesis, the set $S_k = \{t_1, t_2, \ldots, t_k\}$ is linearly independent. Hence, $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$. But this is a contradiction, since we assumed that not all of $\{\alpha_1, \alpha_2, \ldots, \alpha_{k+1}\}$ are zero. Therefore, this case is not possible.

Case 2 : $\alpha_{k+1} \neq 0$

If α_{k+1} is a non-zero field constant, then

$$\begin{aligned} &\alpha_1 t'_1 + \alpha_2 t'_2 + \cdots + \alpha_k t'_k = -\alpha_{k+1} l_1 l_2 \ldots l_k \\ \implies &(\alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_k t_k).l_{k+1} = -\alpha_{k+1} l_1 l_2 \ldots l_k \end{aligned}$$

As it is known that $\mathbb{F}[X]$ is a unique factorization domain ([Gal17]), l_{k+1} has to be an associate of one of $\{l_i\}_{i=1}^{k}$. That is

$$\exists i \quad \text{such that} \quad l_{k+1} = c.l_i, \quad \text{where} \quad c \in \mathbb{F} \setminus \{0\} \quad \text{and} \quad 1 \leq i \leq k.$$

But this contradicts our assumption of the pairwise independence of linear forms $\{l_i\}_{i=1}^{k+1}$.

Hence, S_{k+1} is linearly independent. Therefore, by induction, the claim is verified. If the set of linear forms $\{l_i\}_{i=1}^{n}$ is pairwise independent, then S_n is linearly independent for any $n \geq 2$.

□

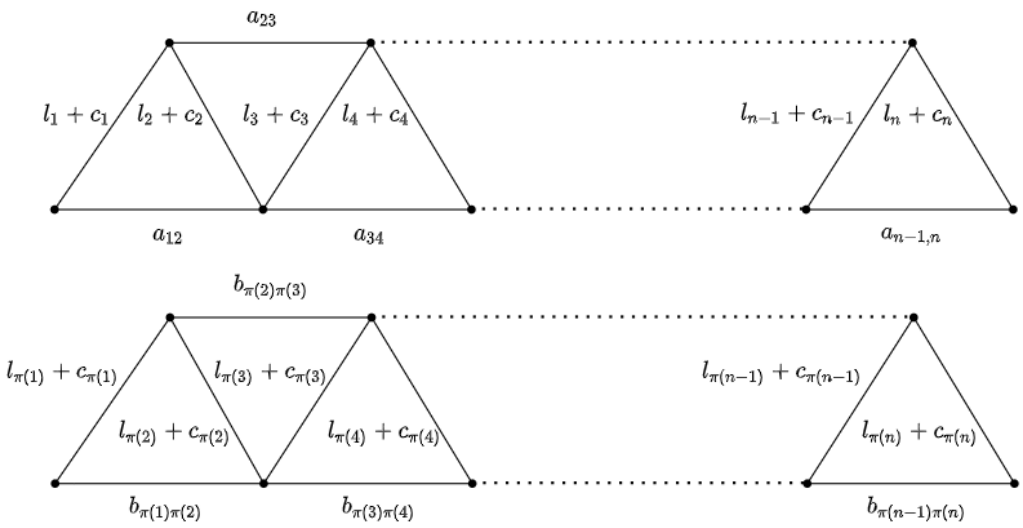
From the previous two lemmas, we arrive at an important result.

Theorem 3.1. *Let A and B be two triangular ABPs with pairwise independent linear forms such that $A - B = x_1x_2 + x_3x_4 + x_5x_6$, then the multi-sets of linear polynomials δ_A and δ_B occurring as diagonal edge weights in A and B respectively are the same.*

Proof. This theorem results from Lemma 3.3 and Lemma 3.4. □

3.2 Triangular ABPs with pairwise independent linear forms

Let us assume that we have two triangular ABPs A and B with pairwise independent linear forms, such that $A - B = x_1x_2 + x_3x_4 + x_5x_6$. From theorem 3.1, we can say that the diagonal edge weights in both the ABPs are the same, but they may exist in different orders. Let the multi-set of linear polynomials which are the diagonal edge weights in both A and B be $\{l_1 + c_1, l_2 + c_2, \ldots, l_n + c_n\}$. Let a_{ij} represent the weight of the edge connecting the diagonal edges with the weights $l_i + c_i$ and $l_j + c_j$ in A. Similarly, let b_{ij} represent the weight of the edge connecting the diagonal edges with the weights $l_i + c_i$ and $l_j + c_j$ in B. Let π be a permutation on $[n]$. In this case, the graphical representations of A and B are:



Although the linear polynomials are the same, they might occur in different orders in A and B. We attempt to find out whether or not they can occur in the same order.

Lemma 3.5. *Let A and B be two triangular ABPs in such a way that $A - B = x_1x_2 + x_3x_4 + x_5x_6$. Let the multi-set of linear polynomials which are the diagonal edge weights in A and B be $\delta = \{l_1 + c_1, l_2 + c_2, \ldots, l_n + c_n\}$. If the set of polynomials $T_n = \{\prod_{1 \leq i \neq j, j+1 \leq n} l_i\}_{j=1}^{n-1}$ is linearly independent, then the linear polynomials occurring as diagonal edge weights in A and B cannot occur in the same order or the reverse order.*

Proof. Let $A_{[d]}$ and $B_{[d]}$ denote the degree d homogeneous parts of the polynomials A and B respectively. Since $A - B = x_1x_2 + x_3x_4 + x_5x_6$,

$$A_{[d]} = B_{[d]}, \quad \forall d \geq 3.$$

From Lemma 3.2, $n \geq 6$. Therefore,

$$A_{[n-2]} = B_{[n-2]},$$

$$\therefore \sum_{j=1}^{n-1} a_{j,j+1} \prod_{1\leq i\neq j,j+1\leq n} l_i + \cancel{\sum_{1\leq i<j\leq n} c_ic_j \prod_{1\leq k\neq i,j\leq n} l_k} =$$

$$\sum_{j=1}^{n-1} b_{\pi(j),\pi(j+1)} \prod_{1\leq i\neq j,j+1\leq n} l_{\pi(i)} + \cancel{\sum_{1\leq i<j\leq n} c_ic_j \prod_{1\leq k\neq i,j\leq n} l_k},$$

$$\therefore \sum_{j=1}^{n-1} a_{j,j+1} \prod_{1\leq i\neq j,j+1\leq n} l_i = \sum_{j=1}^{n-1} b_{\pi(j),\pi(j+1)} \prod_{1\leq i\neq j,j+1\leq n} l_{\pi(i)}. \tag{3.1}$$

Let us assume that the linear polynomials occur in the same order, i.e. π is an identity permutation. Under this scenario, if the set of polynomials $\{\prod_{1\leq i\neq j,j+1\leq n} l_i\}_{j=1}^{n-1}$ is linearly independent, then

$$a_{j,j+1} = b_{j,j+1}, \quad \forall j = 1, 2 \ldots, n-1.$$

This implies that both the ABPs A and B are the same, which in turn would imply that $A - B = 0$. This is a contradiction. Hence, π cannot be an identity permutation and linear polynomials cannot occur in the same order.
We can apply the same argument in the case where the ABPs occur in the reverse order in both the ABPs, i.e. when $\pi(i) = n + 1 - i, \forall 1 \leq i \leq n$.

□

It turns out that $\{\prod_{1\leq i\neq j,j+1\leq n} l_i\}_{j=1}^{n-1}$ is linearly independent in this setup.

Lemma 3.6. *For a triangular ABP containing $n \geq 3$ diagonal edges, if the set of linear forms $\gamma = \{l_i\}_{i=1}^{n}$ occurring in the diagonal edge weights in the ABP is pairwise independent, then the set $T_n = \{\prod_{1\leq i\neq j,j+1\leq n} l_i\}_{j=1}^{n-1}$ is linearly independent.*

Proof. The proof of this lemma proceeds by induction on n.
For the base case of induction where $n = 3$, the set $T_3 = \{l_1, l_3\}$ is linearly independent since l_1 and l_3 are independent.

Now, let us assume the lemma to be true for $n = k$, i.e. $\{l_i\}_{i=1}^{k}$ is pairwise independent and the set T_k is linearly independent. T_k, which contains $k-1$ polynomials of degree $k-2$ each, can be written as:

$$T_k = \{t_1, t_2, \ldots, t_{k-1}\}, \quad \text{where} \quad t_j = \prod_{1 \le i \ne j, j+1 \le k} l_i, \quad \text{for} \quad j = 1, 2, \ldots, (k-1).$$

Now, we proceed towards proving the lemma to be true for $n = k+1$. Let us choose l_{k+1} such that the updated $\gamma = \{l_i\}_{i=1}^{k+1}$ is pairwise independent. Then

$$\begin{aligned} T_{k+1} &= \{t'_1, t'_2, \ldots, t'_{k-1}, l_1 l_2 \ldots l_{k-1}\}, \\ \text{where} \quad t'_j &= \prod_{1 \le i \ne j, j+1 \le k+1} l_i = t_j . l_{k+1}, \quad \text{for} \quad j = 1, 2, \ldots, (k-1). \end{aligned}$$

Let us assume that T_{k+1} is linearly dependent. Therefore, there exist k field constants $\alpha_1, \alpha_2, \ldots, \alpha_k$, not all zero, such that:

$$\alpha_1 t'_1 + \alpha_2 t'_2 + \cdots + \alpha_{k-1} t'_{k-1} + \alpha_k l_1 l_2 \ldots l_{k-1} = 0.$$

Depending upon the value of α_k, we have two possible cases.

Case 1 : $\alpha_k = 0$

If $\alpha_k = 0$, then

$$\begin{aligned} & \alpha_1 t'_1 + \alpha_2 t'_2 + \cdots + \alpha_{k-1} t'_{k-1} = 0 \\ \implies & (\alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_{k-1} t_{k-1}) . l_{k+1} = 0 \end{aligned}$$

By our induction hypothesis, the set $T_k = \{t_1, t_2, \ldots, t_k\}$ is linearly independent. Hence, $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0$. But this is a contradiction, since we assumed that not all of $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ are zero. Therefore, this case is not possible.

Case 2 : $\alpha_k \ne 0$

If α_k is a non-zero field constant,

$$\begin{aligned} & \alpha_1 t'_1 + \alpha_2 t'_2 + \cdots + \alpha_{k-1} t'_{k-1} = -\alpha_k l_1 l_2 \ldots l_{k-1} \\ \implies & (\alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_{k-1} t_{k-1}) . l_{k+1} = -\alpha_{k+1} l_1 l_2 \ldots l_{k-1} \end{aligned}$$

As it is known that $\mathbb{F}[X]$ is a unique factorization domain, l_{k+1} has to be an associate of one of $\{l_i\}_{i=1}^{k-1}$. That is

$$\exists i \quad \text{such that} \quad l_{k+1} = c . l_i, \quad \text{where} \quad c \in \mathbb{F} \setminus \{0\} \quad \text{and} \quad 1 \le i \le k-1.$$

But this contradicts our assumption of the pairwise independence of linear forms $\{l_i\}_{i=1}^{k+1}$. Hence, T_{k+1} is linearly independent.

Therefore, by induction, the lemma is proved. If the linear forms $\{l_i\}_{i=1}^{n}$ are pairwise independent, then T_n is linearly independent for any $n \geq 3$.

□

Lemma 3.5 and Lemma 3.6 lead us to our next key theorem.

Theorem 3.2. *Let A and B be two triangular ABPs with pairwise independent linear forms occurring in the diagonal edge weights such that $A - B = x_1x_2 + x_3x_4 + x_5x_6$, then the linear polynomials occurring as diagonal edge weights in A and B cannot occur in the same order, or the reverse order.*

— 4 —

Conclusion and Future Scope

In this work, we have addressed the question whether the sum of two triangular ABPs can compute $F = x_1x_2 + x_3x_4 + x_5x_6$ or not. In particular, we have shown that if the sum of two triangular ABPs computes F and the linear forms in the diagonal edge weights of both the triangular ABPs are pairwise independent, then the multi-sets of diagonal edge weights in the two triangular ABPs are the same. It is known that if the linear forms occurring in the weights of the diagonal edges in the two triangular ABPs are 3-wise linearly independent, then their sum cannot compute F ([Kor16]). Hence, we analysed the case where the linear forms are pairwise independent and showed that if the diagonal edge weights in both the ABPs occur in the same sequence, then the sum of both the ABPs cannot compute the polynomial F. However, we do not yet know whether some other permutation of diagonal edge weights is possible or not. We would like to know whether the sum of two ABPs with pairwise independent linear forms can compute F or not. Equation (3.1) would hold for any permutation of the linear polynomials in both the ABPs. One way to proceed in this direction is comparing the coefficients of l_i for some $1 \leq i \leq n$ on both sides of the equation. Another approach which might help is to take partial derivatives with respect to a subset of $\{l_i\}_{i=1}^{n}$ on both sides and comparing these partial derivatives. We could further compare the coefficients of l_i for some $1 \leq i \leq n$ in the partial derivatives to gain further insight.

Thereafter, we would like to find out whether the sum of any two triangular ABPs can compute F or not. This would require the analysis of many more cases in which the linear forms in both the ABPs can repeat multiple times. One could approach by finding special restrictions under which both the ABPs turn out to be the same, thereby arriving at a contradiction and concluding that $A - B$ cannot compute F. An alternative way could be to compare the coefficients of higher degree monomials

in both the ABPs. Since $A - B$ computes F, the degree ≥ 3 parts of A and B would cancel each other. Therefore, these higher degree coefficients must be the same. Ultimately, we would like to prove the following conjecture regarding the sum of any constant number of invertible width-2 ABPs:

Conjecture 4.1. $\forall\, c \in \mathbb{N}$, $\exists k(c)$, where $k(c)$ is polynomially large, such that the sum of c width-two invertible ABPs cannot compute $x_1x_2 + x_3x_4 + \ldots + x_{k(c)}x_{k(c)+1}$.

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